

Ex: Compute the tangent line to the curve  
 $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 4\cos(2t) \rangle$  at  $(\sqrt{3}, 1, 2)$ .

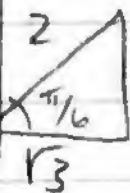
Sol: Tangent function is:

$$\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), -8\sin(2t) \rangle$$

To find the approximate time: solve

$$\vec{r}(t) = \langle \sqrt{3}, 1, 2 \rangle$$

$$\text{ie } \begin{cases} 2\cos(t) = \sqrt{3} \\ 2\sin(t) = 1 \\ 4\cos(2t) = 2 \end{cases} \Rightarrow \begin{cases} \cos(t) = \frac{\sqrt{3}}{2} \\ \sin(t) = \frac{1}{2} \\ \cos(2t) = \frac{1}{2} \end{cases}$$



$$\text{try } t = \frac{\pi}{6} :$$

$$2\cos\left(\frac{\pi}{6}\right) = \sqrt{3} \quad \checkmark$$

$$2\sin\left(\frac{\pi}{6}\right) = 1 \quad \checkmark$$

$$4\cos\left(\frac{2\pi}{6}\right) = 2 \quad \checkmark$$

$\therefore$  the tangent vector  
at  $(\sqrt{3}, 1, 2)$  is

$$\begin{aligned} \vec{r}'\left(\frac{\pi}{6}\right) &= \langle -2\sin\left(\frac{\pi}{6}\right), 2\cos\left(\frac{\pi}{6}\right), -8\sin\left(\frac{2\pi}{6}\right) \rangle \\ &= \langle -2\left(\frac{1}{2}\right), 2\frac{\sqrt{3}}{2}, -8\frac{\sqrt{3}}{2} \rangle \\ &= \langle -1, \sqrt{3}, -4\sqrt{3} \rangle \end{aligned}$$

$\therefore$  the desired tangent line has vector  
equation

$$\begin{aligned} \vec{r}(t) &= \vec{p} + t\vec{r}'\left(\frac{\pi}{6}\right) \\ \vec{r}(t) &= \langle \sqrt{3}, 1, 2 \rangle + t\langle -1, \sqrt{3}, -4\sqrt{3} \rangle \end{aligned}$$

## § 13. ? : Arc Length

Def time: the arc length of a curve  $\vec{R}(t)$  is given between  $t=a$  and  $b$  is given by:

$$S = \int_{t=a}^b |\vec{R}'(t)| dt$$

from Calc II, the arc length was given by ( $\vec{R}(t) = \langle x(t), y(t) \rangle$  on  $a \leq t \leq b$ ):

$$S = \int_{t=a}^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$S = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$S = \int_a^b |\vec{R}'(t)| dt$$

ex) compute the arc length of  $\vec{R}(t) = \langle \cos(t), \sin(t), \ln(\cos(t)) \rangle$  on  $0 \leq t \leq \frac{\pi}{4}$ .

$$S = \int_a^b |\vec{R}'(t)| dt \Rightarrow \int_0^{\pi/4} |\vec{R}'(t)| dt$$

$$\vec{R}'(t) = \left\langle -\sin(t), \cos(t), \frac{-\sin(t)}{\cos(t)} \right\rangle = \langle -\sin(t), \cos(t), -\tan(t) \rangle$$

$$|\vec{R}'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (-\tan(t))^2}$$

$$|\vec{R}'(t)| = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec(t)|$$

positive on interval  
 $\Rightarrow \sec(t)$

$$|\vec{r}'(t)| = \sec(t) \quad 0 \leq t \leq \frac{\pi}{4}$$

$$\therefore S = \int_a^b |\vec{r}'(t)| dt = \int_0^{\pi/4} \sec(t) dt$$

$$S = \left[ \ln |\sec(t) + \tan(t)| \right]_{t=0}^{\pi/4}$$

$$S = \left[ \ln \left| \sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right) \right| \right] - \left[ \ln |\sec(0) + \tan(0)| \right]$$

$$S = \ln |\sqrt{2} + 1| - \underbrace{\ln |1 + 0|}_0$$

$$S = \ln(1 + \sqrt{2})$$

Ex) Compute the arc length of  $\vec{r}(t) = \langle 3\cos t, 3\sin t, t^2 \rangle$  on  $2 \leq t \leq 10$

$$\text{Sol: } S = \int_a^b |\vec{r}'(t)| dt \Rightarrow S = \int_2^{10} |\vec{r}'(t)| dt$$

$$\vec{r}'(t) = \langle -3\sin(t), 3\cos(t), 2t \rangle$$

$$|\vec{r}'(t)| = \sqrt{(-3\cos(t))^2 + (3\sin(t))^2 + (2t)^2} = \sqrt{9(\cos^2 t + \sin^2 t) + 4t^2} = \sqrt{9 + 4t^2}$$

$$\therefore S = \int_2^{10} \sqrt{9 + 4t^2} dt = \int_{t=2}^{10} \frac{3\sqrt{9+4t^2}}{3} \cdot \frac{3 \cdot 2}{2 \cdot 3} dt$$

$$S = \frac{9}{2} \int_2^{10} \sec \theta \sec^2 \theta d\theta$$

$\frac{9 \cdot 4t}{3} = 12t$   
 $\frac{2t}{3} = \tan \theta$   
 $\frac{2}{3} dt = \sec^2 \theta d\theta$   
 $\sqrt{9+4t^2} = \sec \theta$

$$S = \frac{9}{2} \int_2^{10} \sec^3 \theta d\theta$$

to compute  $\int \sec^3 \theta d\theta$ :

$$\begin{aligned} \int \sec^3 \theta d\theta &= \int \sec(\theta) \sec^2(\theta) d\theta = \int \sec \theta (1 + \tan^2 \theta) d\theta \\ &= \int \sec \theta d\theta + \int \sec \theta \tan^2 \theta d\theta \end{aligned}$$

$$= \ln |\sec \theta + \tan \theta| + \int \sec \theta \tan^2 \theta d\theta$$

$$\begin{aligned} u &= \tan \theta & dv &= \sec \theta \tan \theta d\theta \\ du &= \sec^2 \theta d\theta & v &= \sec \theta \end{aligned}$$

$$\int u dv = uv - \int v du$$

$$= \sec \theta \tan \theta - \int \sec \theta \sec^2 \theta d\theta$$

$$= \sec \theta \tan \theta - \int \sec^3 \theta d\theta$$

$$\therefore \int \sec^3 \theta d\theta = \ln |\sec \theta + \tan \theta| + \sec \theta \tan \theta - \int \sec^3 \theta d\theta$$

$$\text{so, } 2 \int \sec^3 \theta = \ln |\sec \theta + \tan \theta| + \sec \theta \tan \theta + C$$

$$\therefore \int \sec^3 \theta d\theta = \frac{1}{2} [\ln |\sec \theta + \tan \theta| + \sec \theta \tan \theta] + C$$

$$\text{hence, } S = \frac{9}{2} \int_2^{10} \sec^3 \theta d\theta$$

$$= \frac{9}{2} \left[ \frac{1}{2} (\ln |\sec \theta + \tan \theta| + \sec \theta \tan \theta) \right]_2^{10}$$

$$S = \frac{9}{4} \left[ \ln \left| \frac{\sqrt{9+4t^2} + \frac{2t}{3}}{3} \right| + \frac{\sqrt{9+4t^2} \cdot \frac{2t}{3}}{3} \right]_{2}^{10}$$

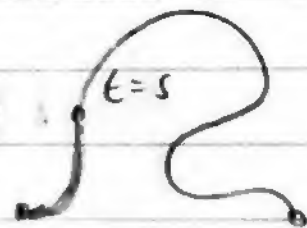
$$S = \frac{9}{4} \left[ \left( \ln \left| \frac{\sqrt{409} + \frac{20}{3}}{3} \right| + \frac{20}{9} \sqrt{409} \right) - \left( \ln \left| \frac{5}{3} + \frac{4}{3} \right| - \frac{5 \cdot 4}{9} \right) \right]$$

$$S = \frac{9}{4} \left[ \ln \left| \frac{\sqrt{409} + 20}{3} \right| + \frac{20\sqrt{409}}{9} - \ln(3) - \frac{20}{9} \right]$$

$$S = 5(\sqrt{409} - 1) + \frac{9}{4} \ln \left| \frac{20 + \sqrt{409}}{9} \right|$$

The arc length of a curve is a natural choice for parameter.

if we would like to parameterize  $\vec{r}(t)$  so that at time  $t=s$ , the arc length (measured from some fixed point) is exactly  $s$ ...



Define the arc length function for a parametrization by:

$$s(\beta) = \int_{t=a}^{\beta} |\vec{r}'(t)| dt$$

"some fixed point"  $\rightarrow$   $t=a$

By FTC,  $s'(\beta) = |\vec{r}'(\beta)|$



Moreover,  $s$  is an increasing function  
provided:  $|r'(p)| \neq 0$  for all  $\beta$ ,  $s$  is  
strictly increasing.

if a function is strictly increasing, it  
has an inverse function.

Next time: this guarantees a unit speed parametrization  
of  $\vec{r}(t)$ .